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A VARIABLE LOAD STEP SOLUTION APPROACH FOR INCREMENTAL TANGENT MODULUS FINITE ELEMENT ANALYSIS

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ENGINEERING MECHANICS DIVISION

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ABSTRACT

This paper addresses the issue of load path discretization in incremental tangent modulus finite element analysis. The view is taken that discretization can best be accomplished during the course of the numerical solution by employing a constraint condition which restricts the level of structural stiffness approximation at each load step. By identifying a field variable which strongly influences the stiffness, a suitable constraint condition can be selected in terms of the nodal variables. With such a constraint, design of a solution algorithm for determining the step size along with the nodal variables is straightforward, as is demonstrated in the text. This variable load step solution approach provides the analyst with a simple yet efficient method for logically controlling step size, without having to resort to time consuming and costly reanalysis procedures to insure that the numerical approximation is within satisfactory tolerance.

Stepwise nonlinear tangent modulus formulations which employ average stiffness matrices for a step are examined in detail. The iterative nature of the solution method is discussed in general terms, while specific consideration is given to the nonhardening Prandtl-Reuss elastic-plastic problem. For this problem the deviatoric stress change which occurs during plastic deformation is identified as the crucial stiffness-governing incremental field variable. Test solutions are presented for a constraint condition which restricts changes in this variable to a specified fraction of the yield stress. The numerical results demonstrate the propriety of this particular condition, and the viability of the variable load step solution approach.

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CONTENTS

A

INTRODUCTION

A common structural analysis problem is to determine the stress and strain distribution throughout a structure at distinct stages of an applied load history. For incremental problems, such as the following elastic-plastic flow theory examples, it is necessary to trace the solution history in discrete steps along the load path to the load states of interest. In this paper we are concerned with accurate methods for discretizing the load path in finite element tangent modulus analysis.

A characteristic of incremental finite element formulations is that the structural stiffness continually changes along the load path, corresponding to the changes which occur in those stress and deformation variables which enter the stiffness definition. Hence, for nontrivial problems, finite load steps imply that the stiffness matrix can never be exact over the duration of a step. The solution at a particular load will always be dependent upon the prior load path discretization, and the analyst must consider means to measure and control this dependency. It is this specific form of numerical error that concerns us here.

Most established finite element computer codes require *a priori* specification of the various load steps of an analysis based upon the best judgment of the analyst. Successive reanalysis is often undertaken with step size refinements according to the observed solution discretization dependency. It is clear that this error control strategy can be very costly and inefficient: it does not directly address the source of the error, and thus in general an acceptable solution can be obtained only in a haphazard fashion.

In the present work the view is taken that discretization error can be best controlled by treating step size as a variable, and determining it numerically along with the incremental nodal variables by employing an error-regulating constraint condition in conjunction with the usual stiffness equilibrium equations. Since the approximate nature of the stiffness is the source of the error, the most proper constraint condition is one which regulates the stiffness approximation level. In the approach taken a particular form of constraint is used which fulfills this function by regulating changes in a field variable which is known to strongly influence the stiffness definition. Although we have limited our development efforts to the design and testing of a solution algorithm for stress analysis of Prandtl-Reuss elastic-plastic materials, the variable load step approach is discussed in general terms since it appears to be applicable to other important nonlinear problems which are routinely treated by the tangent modulus method, such as large deflection analysis.

Since we employ a constraint involving an incremental field variable, regardless of the form that the constraint takes or the specific variable involved, in the assumed displacement finite element method it is in essence always a condition on the nodal displacement change vector $\underline{\Delta U_i}$, for step number $\underline{i}$, and it can be represented in general form as

$$g\ (\underline{\Delta U_i}) = 0. \tag{1}$$

The actual constraint condition chosen for the nonhardening Prandtl-Reuss elastic-plastic examples limits the yield surface deviatoric stress change to a given

fraction of the yield stress. For a large deflection problem it might be proper to constrain a certain displacement gradient distribution. Regardless, with the selection of the constraint condition (1), load path discretization can become an integral part of the numerical procedure. A solution algorithm has been developed for the class of tangent modulus formulations which employ average stiffnesses defined in terms of the undetermined solution, and as a result are stepwise nonlinear. The approach can be most readily appreciated by first considering an elementary stepwise linear formulation which has the stiffness matrix $\underline{K}$ established on the basis of the initial state alone.

For proportional loading to final load vector $\underline{P}$ the stepwise linear problem at step i takes the form

$$\underline{K}\ \underline{\Delta U_i} = \lambda_i \underline{P} \tag{2}$$

where $\lambda_i \underline{P}$ represents the load step, with λ_i being the scalar defining the step size. We wish to select λ_i so that $\underline{\Delta U_i}$ satisfies both (1) and (2). This can be readily accomplished by taking an arbitrary value for λ_i, solving (2) for $\underline{\Delta U_i}$, and then scaling $\underline{\Delta U_i}$ and λ_i by that factor needed to satisfy (1). This advances the solution to the next step and the process is repeated until the final load is reached. In fact, some of the early elastic-plastic tangent modulus formulations[1,2] took exactly this form. Their constraint condition in effect restricted stress increases to control the change of the elastic-plastic boundary. We will discuss the inadequacies of linear formulations along with corrective techniques which lead to stiffness averaging over a step.

We examine then those formulations which employ a stiffness which is dependent upon the yet to be determined vector $\underline{\Delta U_i}$. The equilibrium equation (2) now becomes nonlinear and takes the form

$$K\ (\underline{\Delta U_i})\ \underline{\Delta U_i} = \lambda_i \underline{P}. \tag{3}$$

Most of the established methods use some iterative scheme for solving (3) while holding λ_i fixed. The $\underline{\Delta U_i}$ is obtained by successively solving for trials $\underline{\Delta U_i^j}$ using a stiffness matrix based upon some previous estimate, $\underline{K}\ (\underline{\Delta U_i^{j-1}})$. The variable load step algorithm suggested here scales the solution and λ_i after each iteration until $\underline{\Delta U_i^j}$ is found which satisfies both (1) and (3).

Elastic-plastic test problems were solved using the stepwise nonlinear variable load step algorithm. The convergence properties of the iterative process were found to be essentially the same as the fixed load algorithm. Whereas it is always true that convergence rate slows as stiffness approximation level increases, an important benefit of the variable load step approach is that convergence rate can be regulated *a priori* by adjusting the severity of the constraint condition. The success of the approach depends entirely on the identification of that variable which governs the stiffness approximation. The numerical results show that a proper variable and constraint condition has been chosen for the nonhardening Prandtl-Reuss problem.

1. POPE, G. G. *The Application of the Matrix Displacement Method in Plane Elastic-Plastic Problems* in Proc. Conf. Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, AFFDL-TR-66-80, 1965, p. 635-654.
2. YAMADA, Y., YOSHIMURA, N., and SAKURAI, T. *Plastic Stress-Strain Matrix and its Application for the Solution of Elastic-Plastic Problems by the Finite Element Method.* Int. J. Mech. Sci., v. 10, 1968, p. 343-354.

VARIABLE LOAD STEP SOLUTION ALGORITHM

As we have discussed, once we have decided on a constraint condition (1), a procedure can be developed whereby the given load path is discretized as part of the numerical solution. In this section we describe the algorithm that was developed for stepwise nonlinear tangent modulus formulations having an equilibrium equation of the form (3). We concentrated on the discretization of a particular proportional load segment $\underline{P}$ of the entire load path. The solution history corresponding to any previous loading is assumed known. The algorithm is an extension of the iteration solution process that was used by Marcal and King[3] for fixed load steps.

The Marcal and King procedure involves at each cycle of iteration the formation of a trial structural stiffness matrix followed by the solution of the resulting linear matrix equation. The stiffness trial at the j-th iteration cycle is formed, consistent with the averaging techniques of the formulation, according to an estimated departure $\underline{\Delta U}_i^{j-1}$ from the current structural state, where $\underline{\Delta U}_i^{j-1}$ is the solution of the previous cycle. Hence, at cycle j the governing equilibrium equation takes the form

$$\underline{K}\ (\underline{\Delta U}_i^{j-1})\ \underline{\Delta U}_i^{j} = \lambda_i \underline{P}. \tag{4}$$

The first cycle requires a guess $\underline{\Delta U}_i^{o}$ for definition of the stiffness matrix. Strictly speaking, cycling must continue until successive trial solutions are found to be identical, so that

$$\underline{K}\ (\underline{\Delta U}_i^{j})\ \underline{\Delta U}_i^{j} = \lambda_i \underline{P} \tag{5}$$

and therefore $\underline{\Delta U}_i^{j}$ is then the solution $\underline{\Delta U}_i$ for the step.

In our variable load step approach we adjust λ_i during the course of solution to find that $\underline{\Delta U}_i$ which satisfies both the equilibrium equation (3) and the condition (1). This iterative process starts with an estimate of the load step, λ_i^o, as well as with the guess $\underline{\Delta U}_i^o$. For the first cycle the matrix equation for $\underline{\Delta U}_i^1$ takes the form

$$\underline{K}\ (\underline{\Delta U}_i^{o})\ \underline{\Delta U}_i^{1} = \lambda_i^o \underline{P}. \tag{6}$$

In general $\underline{\Delta U}_i^1$ will not satisfy the constraint condition, although there always is a scalar multiple of this vector which will. The operations required to determine the appropriate scale factor depends upon the nature of the constraint. Regardless, the scale factor is found and the correspondingly scaled displacement solution is used as the trial vector for the next cycle of iteration. The next step size trial follows from interpreting the scale factor as equal to λ_i^1/λ_i^o, as suggested by the linear nature of (6). In general terms, the problem after (6) is solved is to determine λ_i^1 which satisfies

$$g\ (\underline{\Delta U}_i^1 \cdot \lambda_i^1/\lambda_i^o) = 0. \tag{7}$$

3. MARCAL, P. V., and KING, I. P. *Elastic-Plastic Analysis of Two-Dimensional Stress Systems by the Finite Element Method.* Int. J. Mech. Sci., v. 9, 1967, p. 143-155.

The above operations for the first cycle of iteration sets the pattern for subsequent cycles. At cycle j a stiffness matrix is formed according to the estimated displacement $\underline{\Delta U}_i^{j-1} \cdot \lambda_i^{j-1}/\lambda_i^{j-2}$, and a new displacement $\underline{\Delta U}_i^j$ is determined from

$$\underline{K}\ (\underline{\Delta U}_i^{j-1} \cdot \lambda_i^{j-1}/\lambda_i^{j-2})\ \underline{\Delta U}_i^j = \lambda_i^{j-1} \underline{P}. \tag{8}$$

Once $\underline{\Delta U}_i^j$ is obtained, λ_i^j follows from

$$g\ (\underline{\Delta U}_i^j \cdot \lambda_i^j/\lambda_i^{j-1}) = 0. \tag{9}$$

We are of course seeking in this iterative process the step size λ_i and the associated vector $\underline{\Delta U}_i$. Our definition of absolute convergence must be expanded to include the step size parameter. Fortunately we are able to argue the concurrent convergence of both $\underline{\Delta U}_i$ and λ_i. Thus the rate of convergence of the iteration scheme can be conveniently monitored by a test on the cycle-to-cycle change in λ_i^j. The convergence test implemented in this work states that iteration terminates when the relative change in λ_i^j in two successive cycles falls below a given tolerance δ.

We have observed that the stress solution in our elastic-plastic problems is sensitive to the choice of δ. While this is not surprising, it forcefully demonstrates that the level of convergence can govern the viability of the numerical formulation. Again, the issue is most conveniently discussed in terms of the fixed load step solution algorithm. We consider the ramifications of prematurely terminating iteration at a cycle j, taking $\underline{\Delta U}_i^j$ as the solution for the step. Referring to (4), $\underline{\Delta U}_i^j$ can be interpreted as the equilibrium solution for a structure with stiffness $\underline{K}\ (\underline{\Delta U}_i^{j-1})$ under load $\lambda_i \underline{P}$. For our elastic-plastic problems we must choose one of two options for the calculation of the resulting stress changes for the step. The determination of stress involves the use of average constitutive relations based either on $\underline{\Delta U}_i^{j-1}$ or $\underline{\Delta U}_i^j$. Using the first alternative for the calculation will result in a violation of the yield criterion and hardening law when strain hardening occurs. The discussion in the next section provides the background for this statement. The second option will provide a stress solution which satisfies these basic theoretical constraints but is not in equilibrium with the tractions represented by $\lambda_i \underline{P}$. We are unable to provide a quantitative relationship between overall load imbalance and our convergence tolerance parameter δ. However, this is an area which deserves careful consideration for both the variable and fixed load step methods.

To complete our discussion of the variable load step procedure we consider some additional restrictions that should be placed on the allowable magnitude of λ_i. When a definite total load vector $\underline{P}$ is specified there is, of course, the need to restrict $\lambda_i < 1$. Furthermore, $\Sigma\lambda_i$ over all steps must equal unity. The final step to reach the total load $\underline{P}$ will usually be smaller than that allowed by our constraint condition. For this case the algorithm reverts to the standard fixed load procedure. When the load vector $\underline{P}$ is indefinite in the sense that the final magnitude of its components are not specified, then there is no basis for restricting the values of λ_i. This latter case applies to the test problems considered below. There the vector $\underline{P}$ serves merely to specify load direction and the magnitude increases step by step without restriction until limit load is detected.

CONSTRAINT CONDITION FOR ELASTIC-PLASTIC PROBLEMS

The variable load step solution method was implemented for stress analysis of Prandtl-Reuss materials. With this material model yielding is governed by the Mises yield criterion and plastic flow occurs according to the yield surface normality rule. The incremental tangent modulus formulation that was employed has been described by Rice and Tracey.[4] It is useful to begin the discussion by considering the general elastic-plastic problem in the context of various alternate tangent modulus formulations. With this as background, the motivation for our choice of constraint condition can be straightforwardly explained.

In the elastic-plastic problem the structural stiffness changes as the elastic-plastic boundary moves, and as the direction of plastic flow changes at points within the plastic zone. We have mentioned earlier stepwise linear formulations by Pope[1] and Yamada et al.[2] which employ a stiffness based upon the structural solution at the beginning of the step, and restrict load changes so that the evolution of the elastic-plastic boundary is properly traced. In these formulations there is no account taken of the changing constitutive relationships within the plastic zone during the step. Marcal and King[3] devised an average stiffness formulation which alleviates the need to regulate load according to elastic-plastic boundary movement. It was designed to allow an arbitrary step size. Yielding within the step is accommodated by an element stiffness generated from an average constitutive matrix having weighted elastic and elastic-plastic factors. As in the other formulations no account is taken of changing constitutive relationships within the plastic zone and, as a result, the solution may violate the yield criterion and the strain-hardening law.

Rice and Tracey[4] considered the restrictions imposed by the yield criterion on stress changes for a load step. They concentrated on the nonhardening model and proposed an average flow rule, corresponding to a secant approximation to the Mises yield surface, which should be used in defining the stiffness so that the solution will not violate the yield criterion. In the context of the Marcal and King formulation, implementation requires the use of an average matrix for the elastic-plastic portion of the stiffness matrix. For isotropic strain-hardening materials, Tracey[5] extended these considerations further by employing step average hardening rates for satisfaction of a given hardening law.

In each of these latter formulations[3-5] the stiffness is defined in terms of quantities which depend upon the nodal solution. In the notation of the references, the scalar m for weighting the elastic and elastic-plastic factors of the stiffness matrix depends upon $\underline{\Delta U}_i$; the same is true for the average flow vector $\bar{n}$, and the average hardening rate $\bar{Y}'$. Hence these formulations are stepwise nonlinear and have an equilibrium equation which takes the form of (3). Although the averaging methods provide a solution consistent with basic constitutive requirements, the solution nonetheless is approximate. In fact, only if the actual stiffness is constant for the step would the solution be exact. We next consider the issues involved in regulating the level of approximation for the nonhardening

4. RICE, J. R., and TRACEY, D. M. *Computational Fracture Mechanics* in Numerical and Computer Methods in Structural Mechanics, S. J. Fenves et al., ed., Academic Press, New York, 1973, p. 585-623.
5. TRACEY, D. M. *On the Fracture Mechanics Anclysis of Elastic-Plastic Materials Using the Finite Element Method.* Ph.D. Thesis, Brown University, Providence, Rhode Island, 1973.

formulation which employs both the Marcal and King[3] arbitrary load change stiffness averaging, and the average flow rule for yielded points: the formulation discussed by Rice and Tracey.[4]

When discussing the stiffness approximations, it appears that the most important issue to consider is the changing elastic-plastic boundary during an arbitrary load interval. Of course, since the structural stiffness is formed from local stiffnesses, this issue is treated relative to the behavior of distinct material points. In general, a point can experience a number of different stages of deformation during an arbitrary load interval, corresponding to elastic response to yield, followed by various phases of plastic deformation, elastic unloading, and reyielding. The simplest interval history would consist of solely elastic behavior with the stress state always below yield. In this case the local stiffness remains constant. A more involved history is depicted in Figure 1. The deviatoric stress space plot illustrates the important states for a point which begins a load interval below yield at stress $\underline{S}_o$, deforms elastically until the incipient yield state $\underline{S}_1$, and then undergoes continuous elastic-plastic deformation, reaching a final yield surface stress state $\underline{S}_f$. For this case the local stiffness changes abruptly when $\underline{S}_1$ is reached, and thereafter it changes gradually as the stress state follows the yield surface to $\underline{S}_f$. A complex history entailing more than a single stage of plastic deformation with elastic unloadings would be represented in the plot by distinct stress excursions along and inside the yield surface from $\underline{S}_1$ to $\underline{S}_f$.

The Marcal and King average stiffness procedure cannot accurately accomodate a complex interval history. In fact, the procedure is strictly correct (m and $\underline{S}_1$ are exact) only when the displacement rate of change is constant throughout the load interval. The solution history from $\underline{S}_o$ to $\underline{S}_f$ is being approximated by $\underline{\Delta U}_i$ whenever the displacement rate is nonuniform. Hence for complex histories it is clear that totally unsatisfactory approximations can result through the implicit neglect of elastic-plastic phases of the load interval.

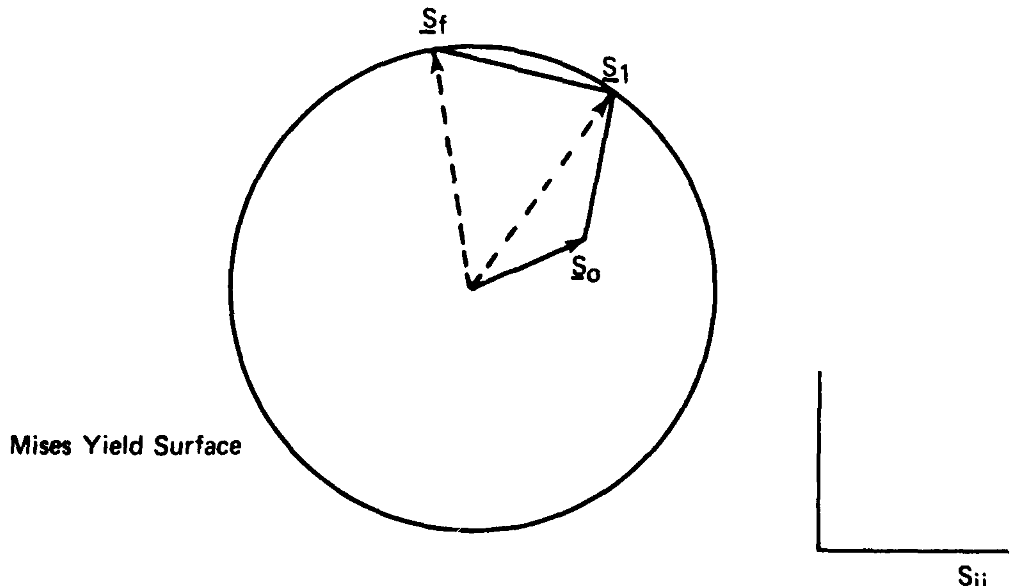


Figure 1. Illustration of deviatoric stress states in example load interval.

From the above it is clear that in incremental elastic-plastic analysis, load steps should be restricted to encompass portions of the structural solution history which involve mildly varying displacement rates in plastically deforming regions. Beyond this there is the further need to restrict load so that the average flow rule reasonably represents the plastic deformation history. This can be accomplished by restricting the stress change $\underline{S}_f-\underline{S}_1$ which occurs during plastic deformation. Yet, perhaps more significant is the fact that by restricting this stress change, displacement rate nonuniformities present within the load step are also regulated. Hence, we have identified a single field variable which directly governs the level of stiffness approximation for our problems.

Our constraint condition follows from the above considerations, and it involves the modulus of $\underline{S}_f-\underline{S}_1$, denoted as ΔS_{sec} below. At each load step we seek that solution which has a maximum ΔS_{sec} value equal to a specified fraction of the yield stress Y. If α represents the specified fraction then (1) takes the form

$$g\ (\Delta \underline{U}_i) = \Delta S_{sec}^{max} - \alpha Y = 0. \qquad (10)$$

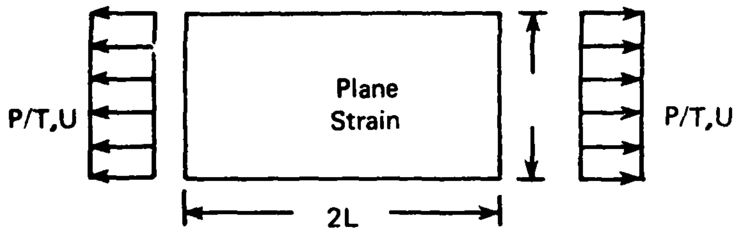
We have not attempted to establish the relationship between structural stiffness approximation level and the constraint parameter α. For the general problem this does not appear to be possible. Krieg and Krieg[6] have considered a related question. They discuss yield surface stress error for homogeneous, constant displacement rate deformation using several constitutive approximations including the average normal. We can conclude only that convergence to the exact solution can be achieved with decreasing α values. The test examples of the next section demonstrate the approach.

NUMERICAL RESULTS

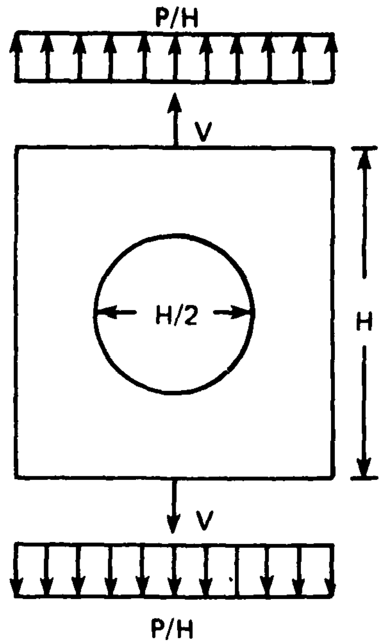
Here we discuss two elastic-plastic problems which were solved using the variable load step approach. The pertinent geometric and loading features of the problems are shown in Figure 2. Following the previous development the nonhardening Prandtl-Reuss constitutive idealization was used, as was the solution constraint condition (10) in conjunction with the tangent modulus formulation described by Rice and Tracey.[4] In the discussion below, Y represents the yield stress in simple tension and E represents the elastic Young's modulus. Solutions were obtained using a Poisson's ratio equal to 0.3.

First we consider a plate in plane strain which is under imposed uniaxial extension. Hill[7] has given the exact solution to this biaxial stress problem. An interesting aspect of the solution is the load-extension relationship from incipient yield to limit load. Whereas only a slight increase in applied tension is possible after yielding and before uncontrolled plastic deformation occurs, the displacement necessary to reach this limit load state is unbounded. Hence

6. KRIEG, R. D., and KRIEG, D. B. *Accuracies of Numerical Solution Methods for the Elastic-Perfectly Plastic Model.* Trans. ASME, J. Pressure Vessel Tech., November 1977, p. 510-515.
7. HILL, R. *The Mathematical Theory of Plasticity.* Claredon Press, Oxford, England, 1950, p. 77-79.



a. Plane strain uniaxial extension elastic-plastic test problem.



b. Plane stress uniaxial tension of elastic-plastic square sheet with circular cutout test problem.

Figure 2.

this provides a valuable test case for our solution approach. There are no spatial variations to contend with in the problem so that a single low-order element adequately models the plate. A four-node isoparametric element was used in the analysis.

In Figure 3 four numerical solutions are given along with the exact solution. The solutions correspond to α values of 0.2, 0.1, 0.05, and 0.025, where α is the freely specified constraint parameter in (10). The results are presented in plots depicting imposed stress/yield stress versus imposed strain/yield strain, (P/T)/Y versus (U/L)/(Y/E). These solutions result regardless of whether P or U is taken as the independent loading parameter. The numerical data, labeled with their associated step numbers, are connected to form piecewise linear approximations to the exact solution which is given by the dashed curve on each plot. As would be expected, the approximations improve, the step sizes decrease, and the number of steps to final load increases as α is reduced. All solutions were generated by specifying a convergence tolerance value δ equal to 10^{-5}. Three cycles were required to meet this convergence test at each step of each solution.

A limit load detection test was employed after each iteration cycle in the load step solution algorithm. If a material point was found which had an equivalent plastic strain increase more than one thousand times greater than its deviatoric elastic strain modulus change, execution was terminated. Such a situation

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A VARIABLE LOAD STEP SOLUTION APPROACH FOR
INCREMENTAL TANGENT MODULUS FINITE ELEMENT
ANALYSIS - Dennis M. Tracey and Colin E. Freese

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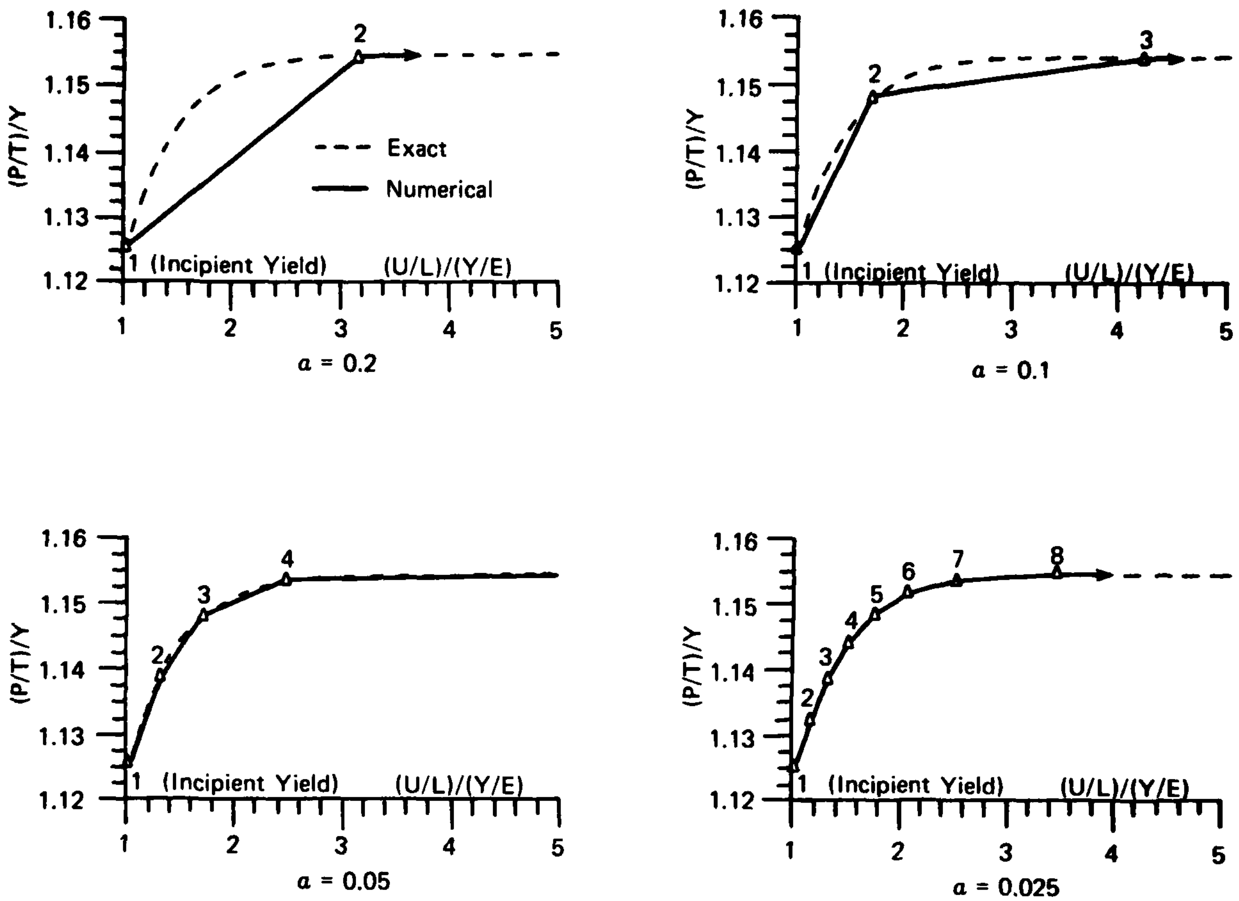

Figure 3. Load-extension results from incipient yield to limit load for plane strain example for α values from 0.2 to 0.025.

would approximate the deformation at limit load which is purely plastic and directed normal to the yield surface at the limiting stress state. As indicated in Figure 3, this test was met at load steps 3, 4, 6, and 9 for the decreasing α values. Furthermore, the test was met in each case at an applied tension very close to the exact limiting value of 1.155Y.

The actual number of steps required for each α choice is predictable for this problem. The constraint parameter restricts deviatoric stress change during plastic deformation to a value equal to αY and for this case the actual change is known to equal 0.185Y. This corresponds to a biaxial stress change from (1.125Y, 0.338Y) at incipient yield to (1.155Y, 0.577Y) at limit load.

This problem served to elucidate an important aspect of numerical solutions near limit load which are obtained using the tangent modulus approach. By virtue of the chosen constraint parameter α (or load step in the fixed load approach) a

solution can be sought and found which places the stress state beyond its exact limiting state. For the present problem this was observed at step 2 of the $\alpha = 0.2$ solution. The resulting biaxial stress state (1.1545Y, 0.5959Y) satisfies the yield criterion, but it represents a point on the yield surface beyond the limit state. This type of solution is achievable because an approximate stiffness is being employed. Nonetheless, there is no insurmountable difficulty with this behavior since the error can be reduced by decreasing α (or step size).

The plot which dramatically illustrates the worth of the variable load step approach is given in Figure 4. The plot gives the $\alpha = 0.025$ discretization results, in the form of step size versus step number, for both the force P and the displacement U loading conditions. Unless one has a detailed knowledge of the exact solution, the discretizations have an unexpected character, suggesting that this problem would entail involved trial and error reanalysis with the standard fixed load approach. The results are displayed relative to P_1 and U_1, the applied force and displacement at incipient yield. The force boundary condition $\Delta P/P_1$ data (marked with triangles) is plotted using the left axis scale. The right axis applies to the displacement boundary condition $\Delta U/U_1$ data (marked with X's). The $\Delta P/P_1$ values vary from 0.0066 for step 2, the first step after incipient yield is reached, to 0.0009 at limit load. The corresponding $\Delta U/U_1$ values range from 0.136 to 0.940.

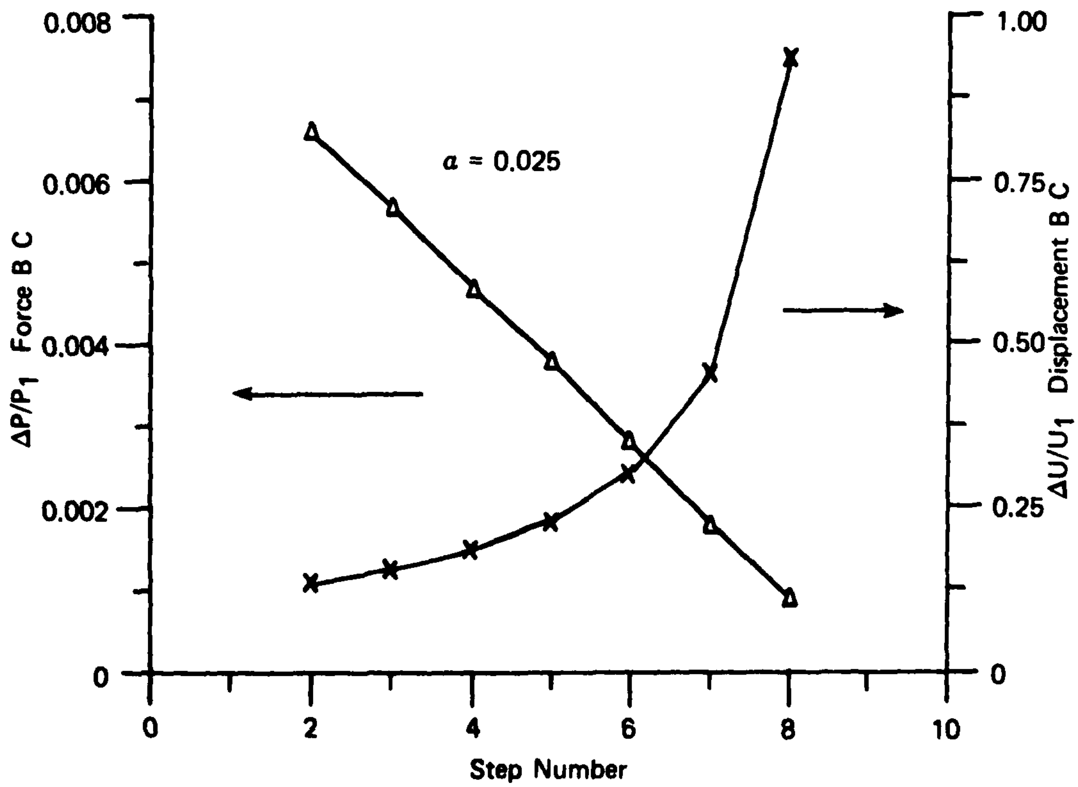

Figure 4. Load discretization results for plane strain problem with constraint parameter α equal to 0.025.

The second problem involves the plane stress uniaxial tension of a square sheet with a centered circular cutout. This problem is more typical of those encountered in practical analysis in the sense that there are undoubtedly complicated spatial variations in the structure which significantly change character as the yielding progresses. However, a comprehensive spatial convergence study was not undertaken, for as throughout this paper the emphasis was placed on load discretization and how it affects the solution for an arbitrary finite element model. Two models consisting of four-node isoparametric elements were employed: one had a 3×6 element arrangement (28 nodes), while the other had a 5×8 (54 node) arrangement.

Load-extension results are given in Figure 5 for α choices of 0.05 and 0.15. The data are plotted in the normalized form (P/H)/Y versus (V/H)/(Y/E), where P/H is the uniform tension across the ends of the plate, and V is the displacement of the center of the loaded edge. The step data are numbered for the α = 0.15 solution and the spread of the plastic zone is illustrated by the shaded elements in

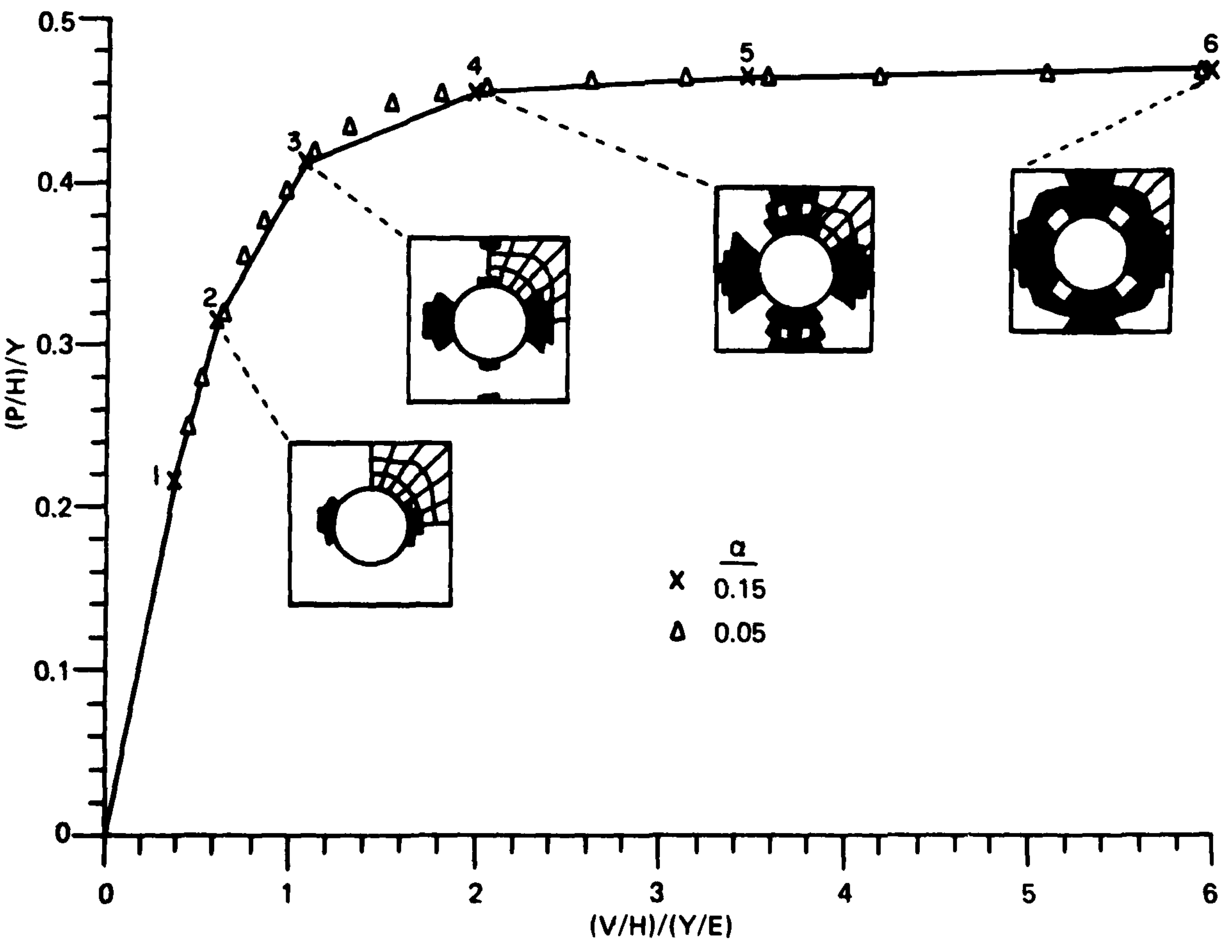

Figure 5. Load-extension results from load free state to limit load for plane stress example for α values of 0.15 and 0.05.

the sketches of the model. As can be seen, the significant differences in the two solutions begin at the knee of the curve when yielding loses its localized character.

Limit load was not detected numerically by the above-mentioned strain test, hence loading was continued until step size was reduced to very small fractions of the incipient yield load P_1. The discretization results are plotted in Figure 6. As in the previous problem it is unlikely that *a priori* judgment would suggest the form of the results, with $\Delta P/P_1$ starting at values of 0.447 and 0.188 and ending at values in the neighborhood of 0.009 and 0.001 for α equal to 0.15 and 0.05, respectively.

A limit analysis of this problem has been presented in the literature.[8] It places the exact limit value of (P/H)/Y between 0.31 and 0.40, whereas our limit load result for α = 0.15 predicts a value of 0.46. A solution using the 5×8 mesh with α = 0.15 was obtained, and it predicts a limit load value of 0.44. This indicates that spatial discretization is the source of the observed differences.

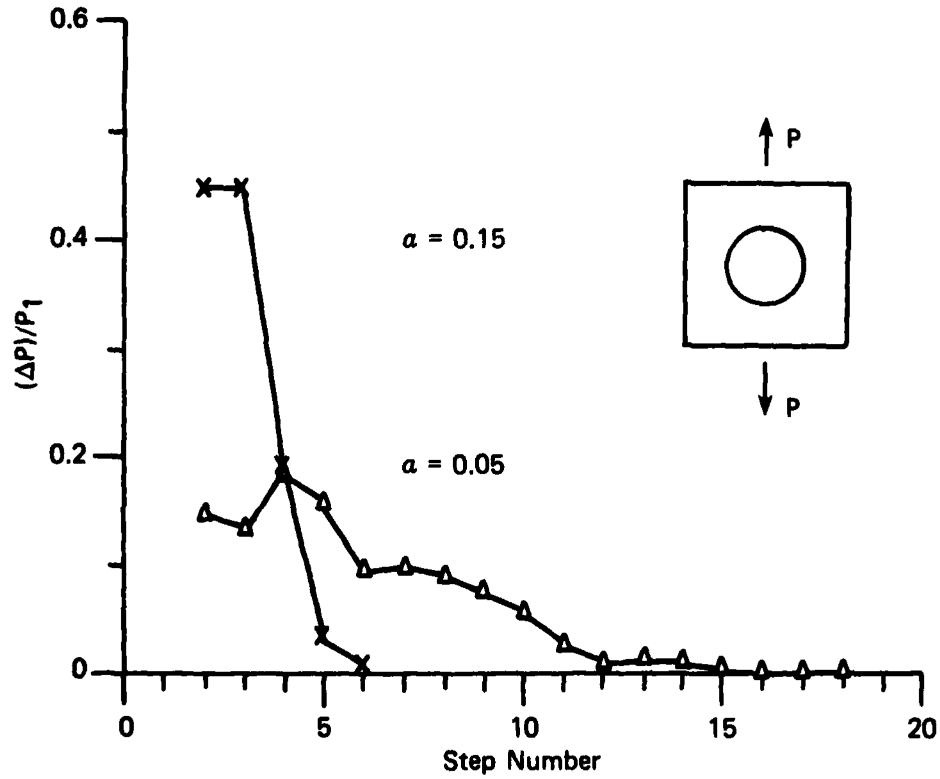

Figure 6. Load discretization results for plane stress problem.

8. GAYDON, F. A., and McCRUM, A. W. *A Theoretical Investigation of the Yield Point Loading of a Square Plate with a Central Circular Hole.* J. Mech. Phys. Solids, v. 2, 1954, p. 156-169.

This problem provided useful data concerning the convergence properties of the solution algorithm at a load step. As would be expected, it was found that the typical number of cycles to meet the δ test increases with α value, and furthermore the required number increases at steps near limit load. A δ value of 10^{-3} was used for this problem. Cycle counts averaged close to 5 for $\alpha = 0.05$, and in the neighborhood of 10 for $\alpha = 0.15$.

As mentioned earlier, the solutions were obtained using a formulation which employs an average yield surface normal[4] to define the plastic flow rule for yielded points. Special considerations were necessary in adapting this averaging technique to the plane stress problem. In plane stress there is the need to establish the average normal in terms of the out-of-plane direct strain increment, but this strain component depends upon the flow rule for its definition. A method was devised for defining these quantities in a way which insures that both the planar stress condition and the yield condition are satisfied. The details of the method will be described in a forthcoming report.

CONCLUSIONS

The numerical results, which now include the load path discretization and corresponding field solution, demonstrate the viability of our variable load step solution algorithm in elastic-plastic analysis. We expect that the algorithm will apply equally well to other nonlinear problems treated by stepwise nonlinear tangent modulus formulations. It is clear that the success of the approach is predicated upon identifying a field variable which controls the level of stiffness approximation, and suitably constraining the variable at each step of the solution. In view of the unpredictable character of the results obtained, the customary fixed load step approach now appears tenuous. The variable step approach not only eliminates the requirement for *a priori* discretization but also provides a series of consistent incremental solutions according to the desired level of approximation. The analyst need only specify the proportional loading segments of the load path and supply values for the constraint parameter (α in our elastic-plastic problems) and the convergence tolerance parameter δ for the iterative solution at each step. With these parameters the analyst can efficiently examine and control solution accuracy both as regards stiffness approximation level and overall satisfaction of equilibrium.

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